1) a) If $n=1$,

then $T=\{x,y\}$ with $x\neq y$

and $S=\{z\}$.

If $f: T\to S$ then

$f(x)=f(y)=z$,

so f is not injective

Now suppose $|S|=n>1$.

$S=\{x_1,\dots,x_n\}$

$T=\{y_1,\dots,y_n,y_{n+1}\}$

Let $\varphi: T \to S$.

Consider $\varphi|_{\{y_1, \dots, y_n\}}$.

If φ maps to a set with less than n elements, then φ is not injective by the induction hypothesis.

So $\varphi(\{y_1, \dots, y_n\}) = S$.

But then $\varphi(y_{n+1}) \in S = \varphi(\{y_1, \dots, y_n\})$

so φ is not injective.

b) Suppose $|S| = n < \infty$.

Let $\varphi : S \to S$ be an injection and suppose φ not surjective. Then by a), φ is not injective, contradiction.

c) $f(x) = e^x$ is

sufficient.

2) The operations will be the usual addition and scalar multiplication restricted to $\mathbb{Q} \times \mathbb{R}$.

Since $\mathbb{R}$ is a field, $(\mathbb{R}, +)$ is an abelian group. Moreover, $(\mathbb{R} \setminus \{0\}, \cdot)$ is also an abelian group, so the unit condition is immediate with 1 as the unit.

Associativity and distributivity of scalar multiplication follow from field distributivity of $\mathbb{R}$ and the fact that $\mathbb{Q} \subseteq \mathbb{R}$. So $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

3) Denote the space in question by $\mathcal{V}$. Then $\mathcal{V}$ is not a field. Let $\mathbb{1}$ denote the function that is constantly 1. Then

$\forall f \in \mathcal{V}$,

$$(\mathbb{1} \cdot f)(x) = 1 \cdot f(x) = f(x),$$

and since the multiplication is commutative, $\mathbb{1}$ is an identity for

$$(\mathcal{V} \setminus 0_{\mathcal{V}}, \cdot).$$

Since field identities are unique, this is the only possible choice for the multiplicative identity. However, if we let $f(x) = x$, then $\forall g \in V$,

$$(f \cdot g)(0) = f(0) \cdot g(0) = 0 \neq 1,$$

so f cannot be invertible.

Therefore, V is not a field.

4) Use the subspace test. Let V denote the vector space of all sequences and let W be the subspace of convergent sequences.

a) $\underline{0_V \in W}$ 0_V is the sequence that is constantly zero, which certainly converges to zero.

b) Suppose $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty} \in W$.

Let $L = \lim_{n\to\infty} a_n$, $M = \lim_{n\to\infty} b_n$

Then by the limit laws,

$$\lim_{n\to\infty} (a_n - b_n)$$
$$= L - M,$$

So $(a_n + b_n)_{n=1}^{\infty} \in W$

c) Let $\alpha \in \mathbb{R}$ and let $(a_n)_{n=1}^{\infty} \in W$. Let $L = \lim_{n \to \infty} a_n$. Then again by the limit laws,

$$\lim_{n \to \infty} (\alpha a_n) = \alpha L,$$

so $(\alpha a_n)_{n=1}^{\infty} \in W$.

Therefore W is a subspace of V.

5) a) The sequence that is constantly zero is in $\ell_\infty(\mathbb{N})$.

The sequence $b_n = n$ is not in $\ell_\infty(\mathbb{N})$.

b) Yes, the space is unchanged.

If $|a_n| < M \ \forall n \in \mathbb{N}$, then certainly $|a_n| \leq M$ $\forall n \in \mathbb{N}$. Now if $|a_n| \leq M \ \forall n \in \mathbb{N}$, then $|a_n|$ is no larger than M, so $|a_n| < M+1$, say.

c) Use the subspace test.

1) $\underline{0_V \in \ell_\infty(\mathbb{N})}$

Since 0_V is again the sequence that is constantly zero, it is in $\ell_\infty(\mathbb{N})$ by a).

2) Let $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty} \in \ell_{\infty}(\mathbb{N})$.

Then $\exists\, M, L > 0$ such that

$|a_n| \leq M \;\; \forall\, n \in \mathbb{N}$ and

$|b_n| \leq L \;\; \forall\, n \in \mathbb{N}$.

Then by the triangle inequality,

$$
\begin{aligned}
|a_n - b_n| &\leq |a_n| + |-b_n| \\
&= |a_n| + |b_n| \\
&\leq M + L,
\end{aligned}
$$

so $(a_n - b_n)_{n=1}^{\infty} \in \ell_{\infty}(\mathbb{N})$

3) Let $(a_n)_{n=1}^{\infty} \in l_\infty(\mathbb{N})$

and $\alpha \in \mathbb{C}$. Then

again, $\exists M > 0$,

$|a_n| \leq M \ \forall n \in \mathbb{N}$.

Then $|\alpha a_n| = |\alpha| |a_n|$

$\leq |\alpha| M$

$\Rightarrow (\alpha a_n)_{n=1}^{\infty} \in l_\infty(\mathbb{N})$.

So $l_\infty(\mathbb{N})$ is a

subspace.

1) a) 4.5   2) 4.5

b) 2

c) 1

3) 4   4) 4

5) a) 2   1 free point

b) 2

c) 5